

## Quantization of the Sphere With Coherent States

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Current views link quantization with dynamics. The reason is that quantum mechanics or quantum field theories address to dynamical systems, i.e., particles or fields. Our point of view here breaks the link between quantization and dynamics: any (classical) physical system can be quantized. Only dynamical systems lead to *dynamical* quantum theories, which appear to result from the quantization of symplectic structures.

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**KEY WORDS:** quantization; sphere; coherent states.

### 1. INTRODUCTION

The procedure developed here, through *coherent states* (CS), allows to quantize any system considered as an *observation set*, i.e., a set of data  $X = \{x\}$ , whose elements can be points, or any kind of parameters. When  $X$  has a symplectic structure, it can be considered as a phase space, and our approach is then equivalent to the usual quantization, although with some peculiar characteristics. But the CS procedure is much more general and can be applied even in the absence of symplectic structure, and in fact of any structure at all (other than a measure) over  $X$ .

A quantization, in this sense, may be considered as a different way to look at the system. It shows numerous analogies with some procedures used in data handling (discussed in more details in Gazeau *et al.*, to appear) for instance those involving wavelets, which are the basic example of coherent states. In many respects, the choice of a quantization appears here as the choice of a resolution to look at the system. As it is well known, some aspects of quantum mechanics may be seen as a noncommutative geometry of the phase space (position and

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momentum operators do not commute). If we quantize a “space of data,” it will be no surprising that a noncommutative geometry emerges. We will show explicitly how a quantization of the ordinary sphere leads to its fuzzy geometry.

## 2. COHERENT STATES

The (classical) system to be quantized is considered as a set of data,  $X = \{x\}$ , with no other specified structure than a measure  $\mu$  (with measure axioms; see Gazeau *et al.*, to appear; Lieb and Loss, 2001). The quantization is defined by the choice of a (separable) Hilbert space  $\mathcal{H}$  with an inclusion map

$$X \ni x \mapsto |x\rangle \in \mathcal{H}. \quad (1)$$

This defines the coherent states  $|x\rangle$ , which must obey two conditions:

1. The resolution of the identity

$$\int_X \mu(dx) |x\rangle\langle x| = \mathbb{I}_{\mathcal{H}}. \quad (2)$$

It implies that the *coherent states*  $|x\rangle$  form an *over-complete* (continuous) basis of  $\mathcal{H}$ .

2. A normalization

$$\langle x|x\rangle = 1. \quad (3)$$

Note that the  $|x\rangle\langle x|$  appear as natural Hermitian operators (orthogonal projection) over  $\mathcal{H}$ .

There exists a natural Hilbert space associated to  $X, \mu$ —the space  $L^2(X, \mu)$  of square-integrable functions over  $X$ . There is an isometric embedding  $W$  of our (closed) Hilbert subspace  $\mathcal{H}$  in  $L^2$ , resulting from the *Weyl-Wigner injection*

$$\mathcal{H} \ni |\psi\rangle \mapsto \Psi \in W(\mathcal{H}) \subset L^2 : x \mapsto \Psi(x) \equiv \langle x|\psi\rangle. \quad (4)$$

Thus, the quantization procedure may also be seen as a peculiar choice of a subspace of  $L^2$ . An explicit procedure is explicated in Gazeau *et al.* (to appear), and applied below to the sphere  $S^2$ . It begins with the selection of a subvector space of  $L^2$  by defining an orthonormal set of  $N$  functions  $\phi_i$ , verifying  $\mathcal{N}(x) \equiv \sum_{i=1}^N |\phi_i(x)|^2 < \infty$ . We note  $\phi_i$ , as a vector, with the ket notation  $|i\rangle$ , and we define the family of coherent states as

$$|x\rangle \equiv \frac{1}{\mathcal{N}(x)} \sum_i \phi_i(x) |i\rangle, \quad (5)$$

which allows to perform the analysis presented above. The resolution of identity implies the existence of a *reproducing kernel*  $K$ , in  $\mathcal{H}$  considered as a subset of

$L^2$ , such that  $\Psi = K \circ \Psi$  (cf. wavelets), i.e.,

$$\Psi(x) = \int \mu(dy)K(x, y)\Psi(y); \quad K(x, y) \equiv \langle x | y \rangle \tag{6}$$

**2.1. Observables and Symbolic Calculus**

A *classical* observable over  $X$  is a function  $f : X \mapsto K(\mathbb{R} \text{ or } \mathbb{C})$ . To any such function  $f$ , we associate the observable over  $\mathcal{H}$ ,

$$A_f \equiv \int_X \mu(dx) f(x)|x\rangle\langle x|. \tag{7}$$

For a large class of observables, these operators are self-adjoint.

The existence of the continuous frame  $\{|x\rangle\}$  enables the definition of a symbolic calculus à la Berezin–Lieb (Berezin, 1975). To each linear, self-adjoint operator (observable)  $\mathcal{O}$  acting on  $\mathcal{H}$ , one associates the *lower (or covariant) symbol*

$$\check{\mathcal{O}}(x) \equiv \langle x | \mathcal{O} | x \rangle \tag{8}$$

and the *upper (or contravariant) symbol* (not necessarily unique)  $\hat{\mathcal{O}}$  such that

$$\mathcal{O} = \int_X \mu(dx) \hat{\mathcal{O}}(x)|x\rangle\langle x|. \tag{9}$$

Note that  $f$  is an upper symbol of  $A_f$ .

They obey the Berezin–Lieb inequalities

$$\int_X \mu(dx) g(\check{\mathcal{O}}(x)) \leq \text{Trg}(\mathcal{O}) \leq \int_X \mu(dx) g(\hat{\mathcal{O}}(x)), \tag{10}$$

where  $g(x)$  is a convex map.

**2.2. First Example: Quantization of the Circle  $\mathcal{S}^1$**

In Gazeau *et al.* (to appear), we gave the simplest examples of application of this procedure: the quantization of a discrete set of elements, and of the unit interval. Here we follow by showing a quantization of the circle  $X = \{\theta\}$ , with the normalized measure  $d\theta/\pi$ .

The simplest possibility is a (real) quantization, with  $\mathcal{H} = \mathbb{R}^2 := \{(x, y)\}$ . The map (1) is defined by

$$X \ni \theta \mapsto |\theta\rangle \equiv (\cos \theta, \sin \theta) \in \mathbb{R}^2. \tag{11}$$

The coherent states  $|\theta\rangle$  are the unit vectors of  $\mathbb{R}^2$  of argument  $\theta$ , which design the unit circle (thus, the embedding). It is easy to check that they form an

over-complete basis of  $\mathcal{H}$ , with the completeness relation (2):

$$\int_0^{2\pi} \frac{d\theta}{\pi} |\theta\rangle\langle\theta| = \mathbb{I}_{\mathcal{H}}.$$

In matrix notation, an observable of  $\mathbb{R}^2$  is written as a linear symmetric  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \frac{a+d}{2}\sigma_0 + \frac{a-d}{2}\sigma_3 + b\sigma_1, \quad (12)$$

so that  $\sigma_0 \equiv \mathbb{I}$  and the Pauli matrices  $\sigma_i$  form a basis for the space of observables. Corresponding upper and lower symbols can be obtained as

$$\begin{aligned} \check{A}(\theta) &= \frac{a+d}{2} + \frac{a-d}{2} \cos 2\theta + b \sin 2\theta, \\ \hat{A}(\theta) &= \frac{a+d}{2} + (a-d) \cos 2\theta + b \sin 2\theta. \end{aligned}$$

### 3. QUANTIZATIONS OF THE 2-SPHERE

#### 3.1. The 2-Sphere

In Gazeau *et al.* (to appear), we proposed a practical method to construct explicitly the coherent states by selecting some peculiar elements of  $L^2$ . Here we apply this method to the quantization of the observation set  $X = S^2$ , the 2-sphere. A point of  $X$  is noted  $x = (\theta, \phi)$ . We adopt the normalized measure  $\mu(dx) = \sin \theta d\theta d\phi/4\pi$ , proportional to the  $SU(2)$ -invariant measure, which is also the surface element.

We know that  $\mu$  is a symplectic form, with the canonical coordinates  $q = \phi$ ,  $p = -\cos \theta$ . This allows to see  $S^2$  as the phase space for the theory of (classical) angular momentum. In this spirit, we will be able to interpret our procedure as the construction of the coherent spin states. Also, our construction will take advantage of the group action of  $SO(3)$  on  $S^2$ .  $S^2$  is embedded in  $\mathbb{R}^3$ , and  $G = SO(3)$  acts as isometry group in  $S^2$ . However, we emphasize again that our quantization procedure is based on the only existence of a measure, and may be used in the absence of metric or symplectic structure.

Quantization is defined by an embedding of  $S^2$  in an Hilbert  $\mathcal{H}$ . This paper deals with the simple case  $\mathcal{H} = \mathbb{C}^2$ . The cases  $\mathcal{H} = \mathbb{R}^n$  will be treated in future works.

### 3.2. Two-Dimensional Hermitian Processing of the 2-Sphere: The Quantum Spin in Its Complex Version

Here, we embed  $S^2$  into the smallest complex Hilbert space possible  $\mathcal{H} = \mathbb{C}^2$ . This quantization leads to the *coherent spin states* (Arecchi *et al.*, 1972; Perelomov, 1986; Radcliffe, 1971). Proceeding as indicated in Gazeau *et al.* (to appear), we define  $\mathcal{H}$  by the selection of the two complex functions

$$\begin{aligned} |1\rangle &\equiv \Phi_1 : \Phi_1(x) = \sqrt{2} \cos \theta/2 & (13) \\ |2\rangle &\equiv \Phi_2 : \Phi_2(x) = \sqrt{2} \sin \theta/2 e^{i\phi}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi. \end{aligned}$$

We define the embedding map

$$x \mapsto |x\rangle = \sqrt{2} \cos \theta/2 |1\rangle + \sqrt{2} \sin \theta/2 e^{i\phi} |2\rangle, \tag{14}$$

leading to

$$\begin{aligned} |x\rangle\langle x| &= 2 \begin{bmatrix} \cos^2 \theta/2 & \cos \theta/2 \sin \theta/2 e^{-i\phi} \\ \cos \theta/2 \sin \theta/2 e^{i\phi} & \sin^2 \theta/2 \end{bmatrix} \\ &= [\sigma_0 + \cos \theta \sigma_3 + \sin \theta \cos \phi \sigma_1 + \sin \theta \sin \phi \sigma_2]. \end{aligned}$$

We can check

$$\int_{S^2} \mu(dx) |x\rangle\langle x| = \mathbb{I}, \quad \langle x|x\rangle = 1.$$

The Pauli matrices  $\sigma_i$  and  $\sigma_0$  form a basis of the  $2 \times 2$  complex hermitian matrices. The upper and lower symbols follow from those of the basis, namely

$$\begin{aligned} \check{\sigma}_0 &= 1 \\ \check{\sigma}_1 &= \sin \theta \cos \phi \\ \check{\sigma}_2 &= \sin \theta \sin \phi \\ \check{\sigma}_3 &= \cos \theta \\ \hat{\sigma}_0 &= 1 \\ \hat{\sigma}_i &= 3\check{\sigma}_i, \quad i = 1, 2, 3. \end{aligned}$$

We obtain easily the operators associated to the functions (coordinates)  $\theta$  and  $\phi$  as

$$A_\theta = \frac{\pi}{8} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \quad A_\phi = \frac{\pi}{4} \begin{bmatrix} 4 & i \\ -i & 4 \end{bmatrix} \tag{15}$$

Their commutator is  $[A_\phi, A_\theta] = \frac{i\pi^2}{64} \sigma_1$ , with

$$\langle x|[A_\phi, A_\theta]|x\rangle = \frac{i\pi^2}{64} \sin \theta \cos \phi \tag{16}$$

and

$$\langle x|[A_\phi, A_\theta]^2|x\rangle = -\frac{\pi^4}{(64)^2}. \tag{17}$$

We may calculate the operators associated to the coordinates in  $\mathbb{R}^3$ :

$$x^1 = \sin \theta \cos \varphi \mapsto \frac{1}{3}\sigma_1, \tag{18}$$

$$x^2 = \sin \theta \sin \varphi \mapsto \frac{1}{3}\sigma_2, \tag{19}$$

$$x^3 = \cos \theta \mapsto \frac{1}{3}\sigma_3, \tag{20}$$

involving the three Pauli matrices. These operators provide the quantum version of the coordinates. We interpret them below in terms of noncommutative geometry.

Note that we can perform the same procedure with the two functions  $\Phi'_1(x) = \sqrt{2} \cos \theta/2 e^{-i\phi/2}$  and  $\Phi'_2(x) = \sqrt{2} \sin \theta/2 e^{-i\phi/2}$ , instead of  $\Phi_1$  and  $\Phi_2$  given by (13), with identical results.

The generalization to  $L + 1$ , instead of 2-dimensions, starts from a choice of  $L + 1$  basis functions (see below), leading to an Hilbert space  $\mathcal{H}$  of dimension  $L + 1$ . As we will see, this is linked to the fuzzy sphere with  $L + 1$  cells.

### 3.3. Link With the Fuzzy Sphere

We recall an usual construction of the fuzzy sphere (Madore, 1995, p. 148). It starts from the decomposition of any smooth function  $f \in C(S^2)$  in spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_m f_{\ell m} Y_m^\ell. \tag{21}$$

We note  $V^\ell$  the  $(2\ell + 1)$ -dimensional vector space generated by the  $Y_m^\ell$ , for fixed  $\ell$ . The direct sum  $\bigoplus_{\ell=0}^L V^\ell$ , generated by the  $Y_m^\ell$  for  $\ell \leq L$ , is a vector space of dimension  $(L + 1)^2$ .

Through the embedding of  $S^2$  in  $\mathbb{R}^3$ , we may write each point of  $S^2$  as  $x = (x^i)$ , with  $\sum_{i=1}^3 (x^i)^2 = 1$ . Any function in  $S^2$  can be seen as the restriction of a function on  $\mathbb{R}^3$ . Moreover, such functions are generated by the homogeneous polynomials in  $\mathbb{R}^3$ . This allows (identifying a function and its restriction) to express equation (21) in a polynomial form in  $\mathbb{R}^3$ :

$$f = f_{(0)} + \sum_{(i)} f_{(i)} x^i + \dots + \sum_{(i_1 i_2 \dots i_\ell)} f_{(i_1 i_2 \dots i_\ell)} x^{i_1} x^{i_2} \dots x^{i_\ell} + \dots, \tag{22}$$

where each sum extends to all symmetric combinations of the  $\ell$  indices to generate  $V^\ell$ . For each fixed value of  $\ell$ , the  $2\ell + 1$  coefficients  $f_{(i_1 i_2 \dots i_\ell)}$  ( $\ell$  fixed) are those of the symmetric traceless  $3 \times 3 \times \dots \times 3$  ( $\ell$  times) matrices.

- To obtain  $S_{\text{fuzzy},L+1}$ , the fuzzy sphere with  $L + 1$  cells, we consider the three generators  $J^i$  of the  $(L + 1)$ -dimensional irreducible unitary representation (IUR) of  $SU(2)$ . They are expressed as  $(L + 1) \times (L + 1)$  matrices obeying

$$[j_i, J_j] = i\epsilon_{ijk} J_k. \tag{23}$$

- To obtain the operator  $F$  associated to any function  $f$ , we first replace each  $x^i$  by the matrix  $X^i = \kappa J^i$ , where  $\kappa = 2r/\sqrt{L^2 + 2L}$ .
- In the above development (22) of  $f$ , we replace each coordinate  $x^i$  by the  $(L + 1) \times (L + 1)$  matrix  $x^i$ , and the usual product by matrix product. Then we truncate the expression obtained at index  $\ell = L$ . These matrices generate the set  $M^{L+1}$  of  $(L + 1)^2$  independent  $(L + 1) \times (L + 1)$  matrices: a closed algebra through the product, which provides a finite approximation to  $C(S^2)$ . According to this construction, a basis of  $M^{L+1}$  is provided by all products of the  $J^i$ 's up to power  $L$ . The corresponding (noncommutative) matrix geometries are finite, fuzzy approximations to the smooth sphere  $S^2$ , which appears as the limit  $N \rightarrow \infty$  of their sequence. Note that  $M^{L+1}$  may be identified to  $\bigoplus_{\ell=0}^L V^\ell$ .

Examples:

- $L = 0$ : we replace the  $x^i$  by the pure number 1 and  $M^1$ , of dimension 1, reduces to  $\mathbb{C}$ .
- $L = 1$ : we replace the  $x^i$  by  $\kappa_1 \sigma^i$ , the three Pauli matrices ( $\kappa_1 = 2r/3$ ). By their products, they generate  $M^2$ , of dimension 4. This gives the geometry of the fuzzy sphere  $S_{\text{fuzzy},2}$  with 2 cells.
- $L = 2$ : we replace the  $x^i$  by  $\kappa_2 J^i$ , with  $\kappa_2 = r/\sqrt{2}$ , and the three rotation matrices;  $[J^i, J^j] = i\epsilon_{ijk} J^k$ . By their products, they generate  $M^3$ , of dimension 9. This gives the geometry of the fuzzy sphere  $S_{\text{fuzzy},3}$  with 3 cells.

According to this construction, the geometry of the fuzzy sphere results from the choice of the algebra  $M^{L+1}$ , of the representation matrices, with their matrix product. This gives the abstract algebra of operators acting on  $S_{\text{fuzzy},L+1}$ . The order  $(L + 1)$  of the matrices invites to see them as acting as the endomorphisms of an Hilbert space of dimension  $(L + 1)$  (Freidel and Krasnov, 2002). This is exactly what provides the coherent states introduced here.

### 3.3.1. Fuzzy Spheres From Coherent States

The CS procedure presented above deals with the case  $L + 1 = 2$ . It associates to the three coordinates  $x^i$ , the three Pauli matrices, i.e., the three operators involved in the construction of  $S_{\text{fuzzy},2}$ . With the identity matrix, they form the vector space of operators  $M^2$ . We introduced them through their action on the Hilbert space  $V^{1/2}$  generated by  $\Phi_1$  and  $\Phi_2$ , which provides a 2-dimensional IUR of  $SU(2)$ .

This suggests the following generalization of the CS procedure which leads to the fuzzy sphere  $S_{\text{fuzzy}, L+1}$ : we consider as the Hilbert space  $V^{L/2}$  which is of the  $(L + 1)$ -dimensional IUR of  $SU(2)$ . When  $L = 2k$  is even, we may chose for  $V^k$  the canonical basis  $|k, i\rangle, i = -k, \dots, k$ , where each  $|k, i\rangle \equiv Y_i^k$  is a spherical harmonic.

This does not apply however when  $L$  is odd. In the general case, we may follow Gazeau and Navaes (2003). We select the basis  $|L/2, i\rangle, i = -L/2, \dots, L/2$ , corresponding to the orthogonal functions  $\Theta_i^{L/2}$ . These functions are defined by the intermediary of the complex variable

$$\begin{aligned} z &\equiv \tan \theta/2 e^{-i\phi}, \text{ as } \Theta_i^{L/2}(x) = \Theta_i^{L/2}(\theta, \phi) \equiv \sqrt{C_{L/2+i}^L} \frac{z^{L/2+i}}{(1 + |z|^2)^{L/2}} \\ &= \sqrt{C_{L/2+i}^L} \cos^{L/2-i} \theta/2 \sin^{L/2+i} \theta/2 e^{-i(L/2+i)\phi}, \end{aligned}$$

with  $C_{L/2+i}^L \equiv \frac{L!}{(L/2+i)!(L/2-i)!}$  (formula (19) of (Gazeau and Novaes, 2003). This allows us to see  $M^{L+1}$  as the set of endomorphisms  $\text{End}(V^{L/2})$ .

The coherent states are constructed following the procedure above:  $|x\rangle = \sum_i \Theta_i^{L/2}(x) |L/2, i\rangle$ . The observables are given by

$$A_f = \sum_{i,j=-L/2}^{L/2} \int \mu(dx) f(x) \bar{\Theta}_i^{L/2}(x) \bar{\Theta}_j^{L/2}(x) |j\rangle \langle i|. \tag{24}$$

In other words,  $[A_f]_{ij} = \int \mu(dx) f(x) \bar{\Theta}_j^{L/2}(x) \Theta_i^{L/2}(x)$ . Now we can develop  $f$  as  $f = \sum_{\ell=0}^{\infty} \sum_m f_{\ell m} \Theta_m^{\ell}$  and calculate the sum. To go further, we take into account the fact that the product  $\bar{\Theta}_i^{L/2}(x) \Theta_j^{L/2}(x)$  can be developed themselves in spherical harmonics, with all terms having  $\ell$  lower than  $L$ . Given the orthogonality of the spherical functions, this implies that the only terms in the development of  $f$  are those with  $\ell \leq L$ . Finally, this leads to

$$F_{ij} = \int \mu(dx) \sum_{\ell=0}^L f_{\ell m} \Theta_m^{\ell}(x) \bar{\Theta}_i^{L/2}(x) \Theta_j^{L/2}(x). \tag{25}$$

Involving the Clebsh–Gordan coefficients

$$C_{mij}^{\ell L/2L/2} \equiv \int \mu(dx) \Theta_m^{\ell}(x) \bar{\Theta}_i^{L/2}(x) \Theta_j^{L/2}(x), \tag{26}$$

we obtain finally

$$F_{ij} = \sum_{\ell=0}^L f_{\ell m} C_{mij}^{\ell L/2L/2}. \tag{27}$$

In particular, the observables  $\hat{Y}_m^{\ell}$  associated to the spherical harmonics  $Y_m^{\ell}, \ell \leq L$ , are in number  $(L + 1)^2$  and provide a basis for  $M^{L+1}$ . They are defined by  $[\hat{Y}_m^{\ell}]_{ij} = C_{mij}^{\ell L/2L/2}$ .



For any value of  $L$ , the CS construction leads to an Hilbert space of dimension  $L + 1$ , as indicated above. What we have shown is that the canonical algebra of operators acting on  $H$  identifies with the algebra of operators acting on  $S_{\text{fuzzy}, L+1}$ , the fuzzy sphere with  $L + 1$  cells.

#### 4. DISCUSSION

The CS quantization method proposed here applies to any observation set. In Gazeau *et al.* (to appear) we applied it to discrete samples and to the unit segment. More general developments will be given in a forthcoming paper. Here we have presented its application to the sphere  $S^2$ . A quantization appear as a choice to look at the sphere with a different point of view, with a finite resolution. We have shown how complex quantizations lead to an explicit construction of the Hilbert space associated to the fuzzy sphere, although we have not examined the noncommutative differential structure associated. We have also emphasized the links with the theory of (irreducible) group representations.

The derivation of coherent spin states shows how this procedure, applied to a symplectic space, is able to give an usual dynamical quantum theory. However, as we claimed, it is much more general, allowing to perform quantization in the absence of any dynamical evolution. In further works, we will examine this possibility and study the application of this quantization procedure to other manifolds, with and without symplectic structure. Potential applications are the derivations of new fuzzy spaces. Also, since quantization can be performed in the absence of any dynamics, this opens perspective for fully covariant approaches, when no time function is defined.

#### REFERENCES

- Arecchi, F. T., Courtens, E., Gilmore, R., and Thomas, H. (1972). *Physical Review A* **6**, 2211.
- Ali, S. T., Antoine, J.-P., and Gazeau, J.-P. (2000). *Coherent States, Wavelets and Their Generalizations*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York.
- Berezin, F. A. (1975). General concept of quantization. *Communications in Mathematical Physics* **40**, 153–174.
- Daubechies, I. (1992). *Ten Lectures on Wavelets*, SIAM-CBMS.
- Feng, D. H., Klauder, J. R., and Strayer, M., eds. (1994). *Coherent States: Past, Present and Future* (Proc. Oak Ridge 1993), World Scientific, Singapore.
- Freidel, L. and Krasnov, K. (2002). *Journal of Mathematical Physics* **43**, 4.
- Gazeau, J.-P., Garidi, T., Huguet, E., Lachièze-Rey, M., and Renaud, J. (to appear), Examples of Berezin-Toeplitz quantization: Finite sets and Unit Interval. In *Proceedings of the Workshop in honor of R. Sharp*, Montreal, 2002, P. Winternitz, ed., CRM-AMS.
- Gazeau, J.-P., and Klauder, J. R. (1999). Coherent states for systems with discrete and continuous spectrum. *Journal of Physics A: Mathematical and General* **32**, 123–132.
- Gazeau, J.-P., and Monceau, P. (2000). Generalized coherent states for arbitrary quantum systems. In *Conférence Moshé Flato 1999—Quantization, Deformations, and Symmetries*, Vol. II, G. Dito and D. Sternheimer, eds., Kluwer, Dordrecht, pp. 131–144.

- Gazeau, J.-P. and Novaes (2003). *Journal of Physics A: Mathematical and General* **36**, 199–212.
- Klauder, J. R. (1963a). Continuous-representation theory, I: Postulates of continuous-representation theory. *Journal of Mathematical Physics* **4**, 1055–1058.
- Klauder, J. R. (1936b). Continuous-representation theory, II: Generalized relation between quantum and classical dynamics. *Journal of Mathematical Physics* **4**, 1058–1073.
- Klauder, J. R. (1996). Coherent states for the hydrogen atom. *Journal of Physics A: Mathematical and General* **29**, L293–298.
- Landau, L. and Lifshitz, E. M. (1958). *Statistical Physics*, Pergamon, New York.
- Lieb, E. H. and Loss, M. (2001). *Analysis Graduate Studies in Mathematics*, 2nd edn., American Mathematical Society, Providence, RI.
- Madore, J. (1995). *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*, Cambridge University Press, Cambridge, UK.
- Magnus, W., Oberhettinger, F., and Soni, R. P. (1996). *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd edn., Springer, Berlin.
- Perelomov, A. M. (1986). *Generalized Coherent States and their Applications*, Springer, Berlin.
- Radcliffe, J. M. (1971). *Some Properties of Coherent Spin States. Journal of Physics A: General Physics* **4**, 313.